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Optimal routing in a problem with constraints and cost functions depending on the task list

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Abstract. A routing problem with precedence conditions and complex cost functions is considered. In this problem, we must choose a starting point, a route (permutation of indices) and a specific trajectory for our process. This trajectory must be consistent with the route. In addition, this route or index permutation defines the sequence of tasks. In addition, the selection of the above route must satisfy the precedence conditions defined by the system of ordered index pairs. These ordered pairs are called address pairs. We consider the additive criterion routing problem. This criterion is natural for the problem of dismantling the system of radiation sources. In this article, we will focus on this engineering problem. In this problem very naturally cost functions arise with a dependency on the list of tasks. Namely, each time the performer touches those and only those sources that were not dismantled at that time. The solution uses widely understood dynamic programming. We build the optimal algorithm for the PC; information about the computational experiment is given.

Introduction

lements of routing arise in various fields of human activity. The report considers a formulation focused on application in nuclear power problems associated with reducing the dose load of performers when carrying out a set of works in conditions of increased radiation. This performance may be related to the actions of the performers in accidents like Chernobyl and Fukushima; another option concerns the task of dismantling a decommissioned NPP power unit. In both cases, we are talking about sequential "switching off" of radiation sources, and this implies routing the movements and the works being performed. In connection with the above-mentioned engineering problems, we note the monographs [1].

Note that mathematical models such as the traveling salesman problem arise in various spheres of human activity [2].

The problem under consideration is a generalization of the well-known traveling salesman problem [3-9]. A distinctive feature of the problem under consideration as compared to the classical problem is the presence of precedence conditions, the dependence of the functional on the list of tasks, and optimization of the starting point.

General notions and designations

We consider a very complicated extreme routing problem. For this, the developed formalization is required. Therefore, we recall some set-theoretical notions and designations. We use standard setteoretical symbolics (quantifiers, connectives and so on). For every object α and β, we use (α,β) for designation of ordered pair (OP) with the first element α and the second element β; here, we follow [10, Ch. II, §3]. If h is an OP, then by $\mathrm{pr}_1(\mathrm{h})$ and $\mathrm{pr}_2(\mathrm{h})$ we define the the first and the second elements of h respectively, $h = (\mathrm{pr}_1(\mathrm{h}), \mathrm{pr}_2(\mathrm{h}))$. As usualy, for every object x, y, and z, we suppose $(x,y,z) \triangleq ((x,y),z)$ ($\triangleq$ is equality by definition). If H is a set, then by $\mathcal{P}'(H)$ we denote the family of all nonempty subsets of H; by $\mathrm{Fin}(H)$ we denote the family of all finite sets of $\mathcal{P}'(H)$. If A and B are nonempry sets, $g\colon A \to B$, and $C \in \mathcal{P}'(\mathrm{A})$, then $g^1(C) \triangleq \{\mathrm{g(x)}\colon \mathrm{x} \in \mathrm{C}\} \in \mathcal{P}'(\mathrm{B})$. Of course, $P \times Q \times R \triangleq (\mathrm{P} \times Q) \times R$ for every nonempty sets P, Q and R.

Let $\mathbb{R}$ be a real line, $\mathbb{R}_+ \triangleq \{\xi \in \mathbb{R} |\ 0 \leq \xi\}$, $\mathbb{N} \triangleq \{1; 2; \ldots\}$, $\mathbb{N}_0 \triangleq \{0; 1; 2; \ldots\}$ and $\overline{p,q} \triangleq \{k \in \mathbb{N}_0 |\ (p \leq k) \& (k \leq q)\}$ with $p \in \mathbb{N}_0$ and $q \in \mathbb{N}_0$. For a nonempty finite set K, we denote by $|K|, |K| \in \mathbb{N}$, the cardinality of K; by $(\mathrm{bi})[K]$ we denote the set of all bijections [12, §3] from $\overline{1,|K|}$ onto K. Let $|\emptyset| \triangleq 0$. If S is a nonempty set, then $\mathcal{R}_+[S]$ is the set of all functions from S into $\mathbb{R}_+$.

Extreme routing problem

In the following, we fix a nonempty set X, a set $X^0 \in \mathrm{Fin}(X)$ (so, X^0 is a nonempty finite subset of X; elements of X^0 play the role of starting points), a number $N \in \mathbb{N}$ for which $N \geq 2$, and nonempty (finite) subsets of X

$$M_1 \in \mathrm{Fin}(X), \ldots, M_N \in \mathrm{Fin}(X). \tag{1}$$

We consider the sets (1) as megacities. Moreover, we fix relations

$$\mathbb{M}_1 \in \mathcal{P}'(M_1 \times M_1), \ldots, \mathbb{M}_N \in \mathcal{P}'(M_N \times M_N). \tag{2}$$

So, with $j \in \overline{1,N}$, in the form of $\mathbb{M}_j$, we have a nonempty subset of $M_j \times M_j$. We consider megacities (1) as objects of visiting. While visiting M_j, the executor implements some works called interior. The specific variant of this works is defined by OP $z = (x,y) \in \mathbb{M}_j$, where x is input point and y is departure point. So, $\mathbb{M}_j$ determines the possible options for performing interior work while visiting M_j.

The sequence of the megacity visiting is defined by index permutation called route. Then $\mathbb{P} = (\mathrm{bi})[\overline{1,N}]$ is the set of all permutations of the index set $\overline{1,N}$. Of course, elements of $\mathbb{P}$ and only they (in the next account) are routes. We consider the following processes

$$x^0 \to (x_1^{(1)} \in M_{\alpha(1)} \rightsquigarrow x_1^{(2)} \in M_{\alpha(1)}) \to \ldots \to (x_N^{(1)} \in M_{\alpha(N)} \rightsquigarrow x_N^{(2)} \in M_{\alpha(N)}), \tag{3}$$

where $x^0 \in X^0$ and $\alpha \in \mathbb{P}$. We must choose $(x^0, \alpha, x_1^{(1)}, x_1^{(2)}, \ldots, x_N^{(1)}, x_N^{(2)})$ for minimization of a criterion. This choice must satisfy some constraints. In addition, x^0 is an element of X^0, $X^0 \subset X$. Moreover,

$$z_1 = (x_1^{(1)}, x_1^{(2)}) \in \mathbb{M}_{\alpha(1)}, \ldots, z_N = (x_N^{(1)}, x_N^{(2)}) \in \mathbb{M}_{\alpha(N)}. \tag{4}$$

In fact, with employment of (3) and (4), we obtain the next variant of our process

$$(x^0, x^0) \to z_1 \in \mathbb{M}_{\alpha(1)} \to \ldots \to z_N \in \mathbb{M}_{\alpha(N)}.$$

Finally, the choice of $\alpha \in \mathbb{P}$ can be constrained by precedence conditions. For introduction of these conditions, we use a set $\mathbf{K} \in \mathcal{P}(\overline{1,N} \times \overline{1,N})$ of OP from elements of $\overline{1,N}$. Elements of $\mathbf{K}$ are called address pairs. For $z \in \mathbf{K}$, the (first) element $\mathrm{pr}_1(z) \in \overline{1,N}$ is called a sender and $\mathrm{pr}_2(z) \in \overline{1,N}$ is called a receiver (of load, information, and so on). The precedence conditions consist in the following: $\alpha \in \mathbb{P}$ is admissible (by precedence conditions) if for every $z = (i,j) \in \mathbf{K}$ a visit to $M_i = M_{\mathrm{pr}_1(z)}$ must precede a visit to M_j. We suppose that

$$\forall \mathbf{K}_0 \in \mathcal{P}'(\mathbf{K}) \ \exists z_0 \in \mathbf{K}_0: \ \mathrm{pr}_1(z_0) \neq \mathrm{pr}_2(z) \quad \forall z \in \mathbf{K}_0. \tag{5}$$

The discussion about this (not restrictive) condition is given in [22]. If $\alpha \in \mathbb{P}$, then $\alpha^{-1} \in \mathbb{P}$ is the permutation inverse to α. Then (under conditions (5))

$$\begin{aligned} \mathbf{A} \triangleq \{\alpha \in \mathbb{P} | \ \forall z \in \mathbf{K} \ \forall t_1 \in \overline{1,N} \ \forall t_2 \in \overline{1,N} \ (z = (\alpha(t_1), \alpha(t_2))) \Rightarrow \\ \Rightarrow (t_1 < t_2)\} = \{\alpha \in \mathbb{P} | \ \alpha^{-1}(\mathrm{pr}_1(z)) < \alpha^{-1}(\mathrm{pr}_2(z)) \ \forall z \in \mathbf{K}\} \in \mathcal{P}'(\mathbb{P}). \end{aligned} \tag{6}$$

So, $\mathbf{A} \neq \emptyset$ and $\mathbf{A} \subset \mathbb{P}$. But, as it is indicated in (3), the choice of $\alpha \in \mathbf{A}$ does not define the process: the notion of trajectory is required. For this, previously we introduce the sets $\mathfrak{M}_j \triangleq \{\mathrm{pr}_1(z): \ z \in \mathbb{M}_j\} \in \mathcal{P}'(M_j)$ (the set of all input points) and

$$\mathbf{M}_j \triangleq \{\mathrm{pr}_2(z): \ z \in \mathbb{M}_j\} \in \mathcal{P}'(M_j) \tag{7}$$

(the set of all departure points), where $j \in \overline{1,N}$. In terms of (7), we introduce the following sets

$$(\mathbb{X} \triangleq X^0 \cup (\cup_{i=1}^{N} \mathfrak{M}_i))\&(\mathbf{X} \triangleq X^0 \cup (\cup_{i=1}^{N} \mathbf{M}_i)). \tag{8}$$

We use the set $\mathbb{X} \times \mathbf{X}$ as phase space of our process (3), (4). Moreover, we introduce the set $\mathfrak{Z}$ of all collections $(z_i)_{i \in \overline{0,N}}$: $\overline{0,N} \to \mathbb{X} \times \mathbf{X}$. Then, with $x \in X^0$ and $\alpha \in \mathbb{P}$, we suppose that

$$\mathcal{Z}_\alpha[x] \triangleq \{(z_t)_{t \in \overline{0,N}} \in \mathfrak{Z} | \ (z_0 = (x, x))\&(z_t \in \mathbb{M}_{\alpha(t)} \ \forall t \in \overline{1,N})\}; \tag{9}$$

of course, $\mathcal{Z}_\alpha[x] \in \mathrm{Fin}(\mathfrak{Z})$. Elements of (9) are trajectories coordinated with route α and starting from point x. Then, for $x \in X^0$, in the form of $\widetilde{\mathbf{D}}[x] \triangleq \{(\alpha, \mathbf{z}) \in \mathbf{A} \times \mathfrak{Z} | \ \mathbf{z} \in \mathcal{Z}_\alpha[x]\} \in \mathrm{Fin}(\mathbf{A} \times \mathfrak{Z})$, we obtain the set of all admissible solutions with the starting point x. Moreover, we consider the routing problem for which $x \in X^0$ may vary for optimization of criterion. For considered routing problem

$$\mathbf{D} \triangleq \{(\alpha, \mathbf{z}, x) \in \mathbf{A} \times \mathfrak{Z} \times X^0 | \ (\alpha, \mathbf{z}) \in \widetilde{\mathbf{D}}[x]\} \in \mathrm{Fin}(\mathbf{A} \times \mathfrak{Z} \times X^0) \tag{10}$$

is the set of all admissible solutions for our more general problem. Such admissible solutions are triplets including route, trajectory, and starting point.

Now, we introduce the corresponding criterion. For this, with $\mathfrak{N} \triangleq \mathcal{P}'(\overline{1,N})$, we suppose that (in general setting)

$$\mathbf{c} \in \mathcal{R}_+[\mathbf{X} \times \mathbb{X} \times \mathfrak{N}], c_1 \in \mathcal{R}_+[\mathbb{X} \times \mathbf{X} \times \mathfrak{N}], \ldots, c_N \in \mathcal{R}_+[\mathbb{X} \times \mathbf{X} \times \mathfrak{N}], f \in \mathcal{R}_+[\mathbf{X}]. \tag{11}$$

In connection with (11), we note that elements of $\mathfrak{N}$ (nonempty subsets of $\overline{1,N}$) play the role of lists of tasks not completed at the current time. Each such list indicates a collection of sources that have not been dismantled at this moment; it is these sources that provide the real radiation exposure on the performer. We use the function $\mathbf{c}$ for estimation of exterior movements, the functions c_j, $j \in \overline{1,N}$, for estimation of interior works, and f — for estimation of terminal state (this is the point $x_N^{(2)}$ in (3)). If $x \in X^0$, $\alpha \in \mathbb{P}$, and $(z_i)_{i \in \overline{0,N}} \in \mathcal{Z}_\alpha[x]$, then

$$\begin{aligned} \mathfrak{C}_\alpha[(z_i)_{i \in \overline{0,N}}] \triangleq \textstyle\sum_{t=1}^{N} [\mathbf{c}(\mathrm{pr}_2(z_{t-1}), \mathrm{pr}_1(z_t), \alpha^1(\overline{t,N})) + \\ + c_{\alpha(t)}(z_t, \alpha^1(\overline{t,N}))] + f(\mathrm{pr}_2(z_N)); \end{aligned} \tag{12}$$

of course, for us, in (12) a variant $\alpha \in \mathbf{A}$ is essential. Then, the general problem can be written in the form

$$\mathfrak{C}_\alpha[\mathbf{z}] \to \min, \quad (\alpha, \mathbf{z}, x) \in \mathbf{D}; \tag{13}$$

the value $V \in \mathbb{R}_+$ of (13) is the smallest of numbers $\mathfrak{C}_\alpha[\mathbf{z}]$, $(\alpha, \mathbf{z}, x) \in \mathbf{D}$. Then, in the form of the set

$$\mathbf{SOL} \triangleq \{(\alpha^0, \mathbf{z}^0, x^0) \in \mathbf{D} |\ \mathfrak{C}_{\alpha^0}[\mathbf{z}^0] = V\} \in \mathrm{Fin}(\mathbf{D}), \tag{14}$$

we obtain (nonempty) set of all optimal solutions of the problem (13). If $x \in X^0$, then we consider the next x-problem

$$\mathfrak{C}_\alpha[\mathbf{z}] \to \min, \quad (\alpha, \mathbf{z}) \in \widetilde{\mathbf{D}}[x]; \tag{15}$$

for (15), the value $\tilde{V}[x] \in \mathbb{R}_+$ of this problem is defined as the smallest of numbers $\mathfrak{C}_\alpha[\mathbf{z}]$, $(\alpha, \mathbf{z}) \in \widetilde{\mathbf{D}}[x]$, and

$$(\mathrm{sol})[x] \triangleq \{(\alpha^0, \mathbf{z}^0) \in \widetilde{\mathbf{D}}[x] |\ \mathfrak{C}_{\alpha^0}[\mathbf{z}^0] = \tilde{V}[x]\} \in \mathrm{Fin}(\widetilde{\mathbf{D}}[x]). \tag{16}$$

So, we select routing problems with fixed starting point. Of course, in our case, the equality

$$V = \min_{x \in X^0} \tilde{V}[x] \tag{17}$$

is realized. We note, that for a point $x^0 \in X^0$ with $\tilde{V}(x^0) = V$ and $(\alpha^0, \mathbf{z}^0) \in (\mathrm{sol})[x^0]$, we obtain that

$$(\alpha^0, \mathbf{z}^0, x^0) \in \mathbf{SOL}. \tag{18}$$

Our goal consists in the determination of the value (17) and a solution from the set (14); we use (18) also. For solving the problem (13), we use DP.

Now, we note some singularities of cost functions (11) for the considered applied problem, connected with dismantling of the radioactive elements. These cost functions are defined here (see (2)) on wider sets than required. Actually, further, the values of the function $\mathbf{c}$ will be needed to estimate the movements between megacities, as well as the movements from $x \in X^0$ to megacities. The values of the functions c_j are essential under conditions when their arguments are chosen from the sets $\mathbb{M}_j \times \mathfrak{N}$. The values $f(x)$ are significant when $x \in \mathbf{M}_j$ for $j \in \overline{1, N}$. For all other cases, the values of $\mathbf{c}, c_1, \ldots, c_N, f$ can be given arbitrarily and, in particular, are proposed to be zero. In this part, we use constructions of [13]. So, in this special variant of our setting, the function $\mathbf{c}$ is defined by [13, (6.1),(6.3),(6.20)]. We recall these constructions of [13] very briefly. Namely, for $x \in \mathbf{X}$, $y \in \mathbb{X}$, and $K \in \mathfrak{N}$, we suppose that

$$\mathbf{c}(x, y, K) = \textstyle\sum_{k \in K} \mathbf{c}(x, y, \{k\}); \tag{19}$$

we sum the values of cost functions for separate sources. If $k \in K$, then $\mathbf{c}(x, y, \{k\})$ is defined by integration of nonlinear function along the trajectory of the executor permutation from x to y. This nonlinear function is defined as dependence inversely proportional to square of the Euclidean distance between executor and the source with index k. This integration procedure is reduced to table formulas; see [13, (6.20)]. So, we obtain the radiation dose obtained by executor during permutation from x to y (see (19)).

As already noted, for $j \in \overline{1, N}$, the values of $c_j(\overline{x}, \overline{y}, \overline{K})$, where $\overline{x} \in \mathbb{X}$, $\overline{y} \in \mathbf{X}$, and $\overline{K} \in \mathfrak{N}$, are essential for $\overline{x} \in \mathfrak{M}_j$, $\overline{y} \in \mathbf{M}_j$, and $j \in \overline{K}$. Now, we discuss only this case. Namely, we sum doses obtained by executor during permutations and work on dismantling of source with index j. In addition, the permutation dose is realized as the sum of two numbers. The first number corresponds to the dose, obtained by executor during permutation from the arrival point $\overline{x}$ to source. During this permutation, we take into account the radioactive influence of all sources with numbers from the set K including source with number j. The second number used in calculation of the permutation dose is defined for trajectory from the source to point of departure $\overline{y}$ (we keep in mind the departure from M_j). In this case, the corresponding dose is defined without the source with index j. So, for this calculation, we take into account the set $K \setminus \{j\}$ of sources. For determination of the certain individual doses for separate sources, we use the integration procedures similar to those discussed in construction of the function $\mathbf{c}$.

Finally, in $c_j(\overline{x},\overline{y},\overline{K})$, the component estimating the immediate work including dismantling is used. We keep in mind the work which is not connected with motion of executor. This component is the certain dose obtained by the executor during the dismantling. This dose depends on the time interval $\vartriangle t_j$ corresponding to the time required for the dismantling realization for source with index j. Moreover, this dose depends on intensity of all sources which are not dismantled at the given time moment. All intensities operate during $\vartriangle t_j$. Of course, the action of source with index j is taken into account since executor is situated near to this source. As a result, $c_j(\overline{x},\overline{y},\overline{K})$ is realized as the sum of all above-mentioned components.

Finally, the function f defines residual radiation dose estimating the executor motion after ending of all works. In particular, the variant $f(x) \equiv 0$ is possible.We suppose that $f\ is$ independent on starting point $x \in X^0$. This is possible when executor is transposed to given fixed point after ending of all works. However, such case is also possible when there are sources that are not included in the initial list for some reason, whose disposal would lead to an unacceptable dose of radiation, and therefore the real plan is limited in this case only to the sources mentioned above. Nevertheless, at the stage of removing the contractor (s) from the operating mode, the mentioned remaining sources affect this contractor (s).

So, we obtain the difficult extremal problem with constraints and complicated cost functions. For solving this problem, we use non-standard dynamic programming (DP) procedure. Therefore, for mathematical setting and the consequent solution construction, the serious formalization is required. For this, the sufficiently detailed summary of mathematical constructions will be given (partially it was made in Section 1 and at the beginning of this Section).

Dynamic programming, 1 (general scheme)

We recall that $\mathfrak{N}$ is the family of all nonempty subsets of $\overline{1,N}$; of course, $\mathfrak{N} \neq \emptyset$. If $K \in \mathfrak{N}$, then we suppose that $\Xi[K] \triangleq \{z \in \mathbf{K} |\ (\mathrm{pr}_1(z) \in K)\&(pr_2(z) \in K)\}$; so, $\Xi[K] \subset \mathbf{K}$ and, for $(i,j) \in \mathbf{K}$

$$((i,j) \in \Xi[K]) \Leftrightarrow ((i \in K)\&(j \in K)).$$

Now, we introduce the special mapping $\mathbf{I}$; so, $\mathbf{I}\colon\ \mathfrak{N} \to \mathfrak{N}$ is defined by the rule

$$\mathbf{I}(\widetilde{K}) \triangleq \widetilde{K}\backslash\{\mathrm{pr}_2(z) : z \in \Xi[\widetilde{K}]\} \quad \forall \widetilde{K} \in \mathfrak{N}. \tag{20}$$

So, for $K \in \mathfrak{N}$, the set $\mathbf{I}(K)$ consists of all indexes $i_0 \in K$ for which $i_0 \neq j$ with every $(i,j) \in \Xi[K]$. Of course, the case $K = \overline{1,N}$ is possible. Then $\mathbf{I}(\overline{1,N}) \neq \emptyset$ and $\mathbf{I}(\overline{1,N}) \subset \overline{1,N}$. In addition, $\Xi[\overline{1,N}] = \mathbf{K}$ and

$$\mathbf{I}(\overline{1,N}) = \overline{1,N}\backslash\{\mathrm{pr}_2(z)\colon\ z \in \mathbf{K}\}.$$

We recall that for $\alpha \in (\mathrm{bi})[K]$ and $m \in \overline{1,|K|}$, in the form of $\alpha^1(\overline{m,|K|})$, we have the set of all indexes $\alpha(j),\ j \in \overline{m,|K|}$; of course, the set $\mathbf{I}(\alpha^1(\overline{m,|K|}))$ is defined by the rule (20), where the stipulation $\widetilde{K} = \alpha^1(\overline{m,|K|}) = \{\alpha(j)\colon\ j \in \overline{m,|K|}\}$ is used. Then, we can introduce admissible (by deletion in sense (20)) routes for visiting megacities $M_k,\ k \in K$. Namely, if $K \in \mathfrak{N}$ then

$$(\mathbf{I} - \mathrm{bi})[K] \triangleq \{\alpha \in (\mathrm{bi})[K] |\ \alpha(m) \in \mathbf{I}(\alpha^1(\overline{m,|K|}))\ \forall m \in \overline{1,|K|}\} \tag{21}$$

is the set of all admissible routes for visiting $M_k,\ k \in K$. Then [14, part 2]

$$\mathbf{A} = (\mathbf{I} - \mathrm{bi})[\overline{1,N}] = \{\alpha \in (\mathrm{bi})[\overline{1,N}] |\ \alpha(m) \in \mathbf{I}(\alpha^1(\overline{m,N}))\ \forall m \in \overline{1,N}\} =$$
$$= \{\alpha \in \mathbb{P} |\ (\alpha(1) \in \mathbf{I}(\overline{1,N}))\&(\alpha(m) \in \mathbf{I}(\overline{1,N}\backslash\alpha^1(\overline{1,m-1}))\ \forall m \in \overline{2,N})\}. \tag{22}$$

So, for complete routes, the precedence admissibility and the deletion admissibility are equivalent. As a result, we have the same supply (22) of valid routes. Let $\widetilde{\mathbb{X}} \triangleq \mathbb{X} \cup \mathbf{X}$. Then, with $K \in \mathfrak{N}$, by $\widetilde{\mathbb{Z}}_K$ we denote the set of all collections

$$(z_i)_{i\in\overline{0,|K|}}\colon\ \overline{0,|K|} \to \widetilde{\mathbb{X}} \times \widetilde{\mathbb{X}}.$$

If $x \in \mathbf{X}$, $K \in \mathfrak{N}$, and $\alpha \in (\mathrm{bi})[K]$, then

$$Z[x;K;\alpha] \triangleq \{(z_i)_{i\in\overline{0,|K|}} \in \widetilde{\mathbb{Z}}_K \mid\ (z_0 = (x,x)) \& \ (z_t \in \mathbb{M}_{\alpha(t)}\ \forall t \in \overline{1,|K|})\} \in \mathrm{Fin}(\widetilde{\mathbb{Z}}_K). \tag{23}$$

Of course, the set (23) is defined with $K = \overline{1,N}$. In addition, $\mathbb{P} = (\mathrm{bi})[\overline{1,N}]$ and $\mathfrak{Z} \subset \widetilde{\mathbb{Z}}_{\overline{1,N}}$; therefore, $Z[x;\overline{1,N};\alpha]$ is defined for $x \in X^0$ and $\alpha \in \mathbb{P}$ (of course, $X^0 \subset \mathbf{X}$). And what is more, with $x \in X^0$ and $\alpha \in \mathbb{P}$

$$\mathcal{Z}_\alpha[x] = Z[x;\overline{1,N};\alpha]. \tag{24}$$

We use (22) and (24) in a totality. Now, we consider the natural definition of the Bellman function. For this, we suppose that

$$v(x,\emptyset) \triangleq f(x) \quad \forall x \in \mathbf{X}. \tag{25}$$

For definition of $v(x,K)$ with $x \in \mathbf{X}$ and $K \in \mathfrak{N}$, we introduce the corresponding value of a partial routing problem. We begin with definition of a criterion. Namely, for $x \in \mathbf{X}$, $K \in \mathfrak{N}$, $\alpha \in (\mathbf{I}-\mathrm{bi})[K]$, and $(z_i)_{i\in\overline{0,|K|}} \in Z[x;K;\alpha]$

$$\mathfrak{C}_\alpha[(z_i)_{i\in\overline{0,|K|}}|K] \triangleq \sum_{t=1}^{|K|} [\mathbf{c}(\mathrm{pr}_2(z_{t-1}), \mathrm{pr}_1(z_t), \alpha^1(\overline{t,|K|})) + c_{\alpha(t)}(z_t, \alpha^1(\overline{t,|K|}))] + f(\mathrm{pr}_2(z_{|K|})). \tag{26}$$

If $x \in \mathbf{X}$ and $K \in \mathfrak{N}$, then

$$D(x,K) \triangleq \{(\alpha,\mathbf{z}) \in (\mathbf{I}-\mathrm{bi})[K] \times \widetilde{\mathbb{Z}}_K \mid \mathbf{z} \in Z[x;K;\alpha]\} \in \mathrm{Fin}((\mathbf{I}-\mathrm{bi})[K] \times \widetilde{\mathbb{Z}}_K). \tag{27}$$

Then, with $x \in \mathbf{X}$ and $K \in \mathfrak{N}$, the next problem

$$\mathfrak{C}_\alpha[(z_i)_{i\in\overline{0,|K|}}|K] \to \min,\ \ (\alpha,(z_i)_{i\in\overline{0,|K|}}) \in D(x,K), \tag{28}$$

is defined (see (26), (27)). We associate to $x \in \mathbf{X}$ (see (8)) and $K \in \mathfrak{N}$ the value

$$v(x,K) \triangleq \min_{(\alpha,\mathbf{z})\in D(x,K)} \mathfrak{C}_\alpha[\mathbf{z}|K] \in \mathbb{R}_+ \tag{29}$$

of the problem (28). Since $\mathcal{P}(\overline{1,N}) = \mathfrak{N} \cup \{\emptyset\}$, by (25) and (29) the Bellmann function $v \in \mathcal{R}_+[\mathbf{X} \times \mathcal{P}(\overline{1,N})]$ is defined. We note the obvious connection (29) and the function $\tilde{V}[\cdot] \triangleq (\tilde{V}[x])_{x\in X^0} \in \mathcal{R}_+[X^0]$. Namely, by (10), (21), (24), and (27) we have the equality

$$\widetilde{D}[x] = D(x,\overline{1,N}) \quad \forall x \in X^0. \tag{30}$$

Moreover, with employment of (21) and (24), we obtain (see (13) and (26)) that, with $x \in X^0$, $\alpha \in \mathbf{A}$, and $\mathbf{z} \in \mathcal{Z}_\alpha[x]$

$$\mathfrak{C}_\alpha[\mathbf{z}] = \mathfrak{C}_\alpha[\mathbf{z}|\overline{1,N}] \tag{31}$$

(we use the obvious equality $|\overline{1,N}| = N$). Then, using (29), (30) and (31), we obtain that

$$\tilde{V}[x] = v(x,\overline{1,N}) \quad \forall x \in X^0. \tag{32}$$

So, in fact, $\tilde{V}[\cdot]$ is constriction of v to X^0; more precisely, $\tilde{V}$ is the constriction of $v(\cdot,\overline{1,N})$ to X^0. We note that, for $K \in \mathfrak{N}$, $j \in \mathbf{I}(K)$ and $z \in \mathbb{M}_j$, we obtain $\mathrm{pr}_2(z) \in \mathbf{M}_j$ and, in particular (see (8)), $\mathrm{pr}_2(z) \in \mathbf{X}$; as a result, $v(\mathrm{pr}_2(z), K\setminus\{j\})$ is defined.

Proposition 1 (see [16]). If $x \in \mathbf{X}$ and $K \in \mathfrak{N}$, then

$$v(x,K) = \min_{j\in\mathbf{I}(K)}\min_{z\in\mathbb{M}_j}[\mathbf{c}(x,\mathrm{pr}_1(z),K) + c_j(z,K) + v(\mathrm{pr}_2(z),K\setminus\{j\})]. \quad (33)$$

From Proposition 1, the next equality follows: at $x \in X^0$,

$$\tilde{V}[x] = \min_{j\in\mathbf{I}(\overline{1,N})}\min_{z\in\mathbb{M}_j}\left[\mathbf{c}\left(x,\mathrm{pr}_1(z),\overline{1,N}\right) + c_j\left(z,\overline{1,N}\right) + v\left(\mathrm{pr}_2(z),\overline{1,N}\setminus\{j\}\right)\right]. \quad (34)$$

Of course, the construction of function v is connected with serious difficulties of the computing nature. But, rational employment of the precedence conditions gives some new possibilities in this direction (we mean the lowering of the computing complexity). These possibilities are considered in the next section.

Dynamic programming, 2 (economical option)

In our section, we consider the approach of [13–15] connected with construction and employment of the Bellman function layers. For this, at first, we introduce the set

$$\mathcal{G} \triangleq \{K \in \mathfrak{N}|\ \forall z \in \mathbf{K}\ (\mathrm{pr}_1(z) \in K) \Rightarrow (\mathrm{pr}_2(z) \in K)\}. \quad (35)$$

Moreover, let $\mathcal{G}_s \triangleq \{K \in \mathcal{G}|\ s = |K|\}\ \ \forall s \in \overline{1,N}$. In addition, $\mathcal{G}_N = \{\overline{1,N}\}$ and $\mathcal{G}_1 = \left\{\{t\}\colon\ t \in \overline{1,N}\setminus\mathbf{K}_1\right\}$, where $\mathbf{K}_1 = \{\mathrm{pr}_1(z)\colon\ z \in \mathbf{K}\}$. Moreover, by [16, (4.6)]

$$\mathcal{G}_{s-1} = \{K\setminus\{t\}\colon K \in \mathcal{G}_s,\ t \in \mathbf{I}(K)\}\ \ \forall s \in \overline{2,N}. \quad (36)$$

So, by (36) the recurrent procedure is defined: $\mathcal{G}_N$ is known and every transformation for families $\mathcal{G}_s$ is realized by (36).

The layers of the position space. We consider (x,K), where $x \in X$ and $K \in \mathfrak{N}$, as a position. In the position space, we construct the sets $D_0, D_1,\ldots,D_N$. In addition, D_0 and D_N are defined very simply. Namely, $D_0 \triangleq \{(x,\emptyset)\colon\ x \in \widetilde{\mathcal{M}}\}$ with

$$\widetilde{\mathcal{M}} \triangleq \bigcup_{i\in\overline{1,N}\setminus\mathbf{K}_1} \mathbf{M}_i.$$

Moreover, $D_N \triangleq \left\{\left(x,\overline{1,N}\right)\colon\ x \in X^0\right\}$. For construction of intermediate layers, the special procedure is used. Namely, for $s \in \overline{1,N-1}$, we introduce sequentially for $K \in \mathcal{G}_s$ that

$$J_s(K) \triangleq \{j \in \overline{1,N}\setminus K|\ \{j\}\cup K \in \mathcal{G}_{s+1}\};\ \ \mathcal{M}_s[K] \triangleq \bigcup_{j\in J_s(K)} \mathbf{M}_j;\ \ \mathbb{D}_s[K] \triangleq \{(x,K)\colon\ x \in \mathcal{M}_s[K]\},$$

we consider the last set as a cell of the position space. Next, we suppose that with $s \in \overline{1,N-1}$

$$D_s \triangleq \textstyle\bigcup_{K\in\mathcal{G}_s} \mathbb{D}_s[K]. \quad (37)$$

So, D_i has a cell-like structure (i = 1,…N). As a result, we obtain

$$D_0 \neq \emptyset,\ D_1 \neq \emptyset,\ldots,D_N \neq \emptyset, \quad (38)$$

where $D_j \subset \mathbf{X}\times\mathcal{P}(\overline{1,N})\quad \forall j \in \overline{0,N}$. We note that (see [15, Section 4])

$$(\mathrm{pr}_2(z),K\setminus\{j\}) \in D_{s-1}\ \forall s \in \overline{1,N}\ \ \forall (x,K) \in D_s\ \forall j \in \mathbf{I}(K)\ \ \forall z \in \mathbb{M}_j. \quad (39)$$

The layers of Bellman function. Now, we use (37). So, for $l \in \overline{0, N}$, we introduce $v_l \in \mathcal{R}_+[D_l]$ by the following rule

$$v_l(x, K) \triangleq v(x, K) \ \forall (x, K) \in D_l. \tag{40}$$

Then, $v_0 \in \mathcal{R}_+[D_0]$ is defined by the values $v(x, \emptyset),\ x \in \widetilde{\mathcal{M}}$, where $\widetilde{\mathcal{M}} \subset \mathbf{X}$. Then, by (25)

$$v_0(x, \emptyset) = f(x) \ \forall x \in \widetilde{\mathcal{M}}. \tag{41}$$

With employment of (39), we obtain the following property: if $s \in \overline{1, N}$, $(x, K) \in D_s$, $j \in \mathbf{I}(K)$, and $z \in \mathbb{M}_j$, then the value $v_{s-1}(\mathrm{pr}_2(z), K \setminus \{j\}) \in \mathbb{R}_+$ is defined correctly. In addition, by Proposition 1 we have the next property: if $s \in \overline{1, N}$, then transformation $v_{s-1} \to v_s$ is defined by the rule

$$\begin{aligned} v_s(x, K) = {} & \min_{j \in \mathbf{I}(K)} \min_{z \in \mathbb{M}_j} [\mathbf{c}(x, \mathrm{pr}_1(z), K) + c_j(z, K) + v_{s-1}(\mathrm{pr}_2(z), K \setminus \{j\})] \\ & \forall (x, K) \in D_s. \end{aligned} \tag{42}$$

So, we obtain the following recurrent procedure

$$v_0 \to v_1 \to \ldots \to v_N, \tag{43}$$

where v_0 is defined by (41) and, for $s \in \overline{1, N}$, the transformation v_{s-1} to v_s is realized by (42).

We note that by (40)

$$\tilde{V}[x] = v_N(x, \overline{1, N}) \quad \forall x \in X^0 \tag{44}$$

(we use the above-mentioned representation of D_N). From (42)–(44), the important property of our DP procedure follows: this procedure (42)–(44) is universal with respect to $x \in X^0$. Namely, dependence on the initial state from X^0 arises only at the last stage of (43). This dependence is defined by (44).

Now, we note two possible variants of employment for procedure (43).

1) Algorithm for determination of the global extremum. In this part, we consider an analog of the scheme of [16-18]. Namely, we use the variant of (43) with the layers overwriting. For this variant, in the computer memory, at every stage of the procedure, only one of functions from (43) is situated. This permits us to economize the memory resources. Namely, the function v_0 is defined by (41). In addition, in fact, v_0 is the required "part" of the terminal function f. In connection with employment (42), we note the following possibility. If $s \in \overline{1, N-1}$ and the function v_{s-1} is already constructed, then we determine v_s by (42). Then, v_{s-1} is replaced by v_s; the function v_{s-1} is destroyed. These process of realization of the Bellman function layers continues until exhaustion of the index set $\overline{1, N}$. As a result, we obtain the function $v_N \in \mathcal{R}_+[D_N]$. Therefore, we obtain the values $\tilde{V}[x] = v_N(x, \overline{1, N})$, where $x \in X^0$ (we recall that $(x, \overline{1, N}) \in D_N$ with $x \in X^0$ and the values $v_N(x, \overline{1, N}) \in \mathbb{R}_+$ are already defined). Now, for the determination of V, we use (17). The obtained value V can be used for justified prediction of the process quality and for possible heuristics testing.

2) Algorithm for constructing the optimal solution. In this part, we consider the procedure of construction of some solution from the set (14). Namely, we suppose that all functions (43) are already constructed. In addition, the procedure of finding of the functions (43) is constructed on basis of (42). In this part, we suppose that all functions (43) are saved in the computer memory. So, $v_0, v_1, \ldots, v_N$ are known. Then, by (44) we have all values $\tilde{V}[x],\ x \in X^0$. Then, we solve the problem

$$\tilde{V}[x] \to \min, \quad x \in X^0. \tag{45}$$

So, we find $x^0 \in X^0$ for which $\tilde{V}[x^0] = V$ (see (17)). Now, we consider the problem (15) at $x = x^0$. In addition, we construct an element of $(\mathrm{sol})[x^0]$ (see (16)) by the DP procedure.

Namely, suppose that $\mathbf{z}^{(0)} \triangleq (x^0, x^0)$. Then, by (42) and (44)

$$V = \tilde{V}[x^0] = v_N(x^0, \overline{1,N}) = \min_{j\in\mathbf{I}(\overline{1,N})} \min_{z\in\mathbb{M}_j} [\mathbf{c}(x^0, \mathrm{pr}_1(z), \overline{1,N}) + \\ + c_j(z, \overline{1,N}) + v_{N-1}(\mathrm{pr}_2(z), \overline{1,N}\setminus\{j\})]. \quad (46)$$

In connection with (46), we solve the problem

$$\mathbf{c}(x^0, \mathrm{pr}_1(z), \overline{1,N}) + c_j(z, \overline{1,N}) + v_{N-1}(\mathrm{pr}_2(z), \overline{1,N}\setminus\{j\}) \to \min, \\ j \in \mathbf{I}(\overline{1,N}),\ z \in \mathbb{M}_j. \quad (47)$$

We find $\eta_1 \in \mathbf{I}(\overline{1,N})$ and $\mathbf{z}^{(1)} \in \mathbb{M}_{\eta_1}$ as a solution for the problem (47). So

$$\mathbf{c}(x^0, \mathrm{pr}_1(\mathbf{z}^{(1)}), \overline{1,N}) + c_{\eta_1}(\mathbf{z}^{(1)}, \overline{1,N}) + v_{N-1}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \overline{1,N}\setminus\{\eta_1\}) = V \quad (48)$$

(we use (46)). In addition, by (39)

$$(\mathrm{pr}_2(\mathbf{z}^{(1)}), \overline{1,N}\setminus\{\eta_1\}) \in D_{N-1} \quad (49)$$

(in (49), we take into account that $(x^0, \overline{1,N}) \in D_N$). Therefore, by (42) and (49)

$$v_{N-1}\big(\mathrm{pr}_2(\mathbf{z}^{(1)}), \overline{1,N}\setminus\{\eta_1\}\big) = \min_{j\in\mathbf{I}(\overline{1,N}\setminus\{\eta_1\})} \min_{z\in\mathbb{M}_j} [\mathbf{c}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \mathrm{pr}_1(z), \overline{1,N}\setminus\{\eta_1\}) + \\ + c_j(z, \overline{1,N}\setminus\{\eta_1\}) + v_{N-2}(\mathrm{pr}_2(z), \overline{1,N}\setminus\{\eta_1; j\})]. \quad (50)$$

In connection with (50), we consider the next problem:

$$\mathbf{c}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \mathrm{pr}_1(z), \overline{1,N}\setminus\{\eta_1\}) + c_j(z, \overline{1,N}\setminus\{\eta_1\}) + \\ + v_{N-2}(\mathrm{pr}_2(z), \overline{1,N}\setminus\{\eta_1; j\}) \to \min, \quad j \in \mathbf{I}(\overline{1,N}\setminus\{\eta_1\}),\ z \in \mathbb{M}_j. \quad (51)$$

We find $\eta_2 \in \mathbf{I}(\overline{1,N}\setminus\{\eta_1\})$ and $\mathbf{z}^{(2)} \in \mathbb{M}_{\eta_2}$ as a solution for the problem (51):

$$v_{N-1}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \overline{1,N}\setminus\{\eta_1\}) = \mathbf{c}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \mathrm{pr}_1(\mathbf{z}^{(2)}), \overline{1,N}\setminus\{\eta_1\}) + \\ + c_{\eta_2}(\mathbf{z}^{(2)}, \overline{1,N}\setminus\{\eta_1\}) + v_{N-2}(\mathrm{pr}_2(\mathbf{z}^{(2)}), \overline{1,N}\setminus\{\eta_1; \eta_2\}). \quad (52)$$

Of course, the next inclusion takes place:

$$(\mathrm{pr}_2(\mathbf{z}^{(2)}), \overline{1,N}\setminus\{\eta_1; \eta_2\}) \in D_{N-2} \quad (53)$$

(in connection with (53), we recall (39)). From (48) and (52), we obtain that

$$V = \mathbf{c}(x^0, \mathrm{pr}_1(\mathbf{z}^{(1)}), \overline{1,N}) + \mathbf{c}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \mathrm{pr}_1(\mathbf{z}^{(2)}), \overline{1,N}\setminus\{\eta_1\}) + \\ + c_{\eta_1}(\mathbf{z}^{(1)}, \overline{1,N}) + c_{\eta_2}(\mathbf{z}^{(2)}, \overline{1,N}\setminus\{\eta_1\}) + v_{N-2}(\mathrm{pr}_2(\mathbf{z}^{(2)}), \overline{1,N}\setminus\{\eta_1; \eta_2\}). \quad (54)$$

Remark 1. If $N = 2$, then by (54) the pair $((\eta_i)_{i\in\overline{1,2}}, (\mathbf{z}^{(i)})_{i\in\overline{0,2}})$ is an element of $(\mathrm{sol})[x^0]$ (the corresponding proof is very simple).

Returning to general case of N, $N \geq 2$, we note that the procedures similar to (47) and (51) must continue until exhaustion of the index set $\overline{1,N}$. By these procedures an admissible solution $(\eta, (\mathbf{z}^{(j)})_{j\in\overline{0,N}}) \in \tilde{\mathbf{D}}[x^0]$, $\eta = (\eta_j)_{j\in\overline{1,N}} \in \mathbf{A}$, with $\mathfrak{C}_\eta[(\mathbf{z}^{(j)})_{j\in\overline{0,N}}] = \tilde{V}[x^0]$ will be constructed. Of course, by (17) $(\eta, (\mathbf{z}^{(j)})_{j\in\overline{0,N}}) \in (\mathrm{sol})[x^0]$ and by the choice of x^0 $\mathfrak{C}_\eta[(\mathbf{z}^{(j)})_{j\in\overline{0,N}}] = V$. Therefore, we obtain that $(\eta, (\mathbf{z}^{(j)})_{j\in\overline{0,N}}, x^0) \in \mathbf{SOL}$. So, optimal solution of the problem (13) is constructed.

Our theoretical scheme was realized as algorithm in PC. Now, we note only results for one planar example. Namely, optimal solution was constructed in the case when $N = 31, |M_i| \equiv 20, |K| = 34$, $|X^0| = 6$, the computing time about 12 hours.

Conclusion

For an extreme routing problem focused on engineering applications, we have built a solution method using a non-standard dynamic programming option. On the basis of this method, an optimal algorithm has been developed and implemented on a personal computer. The statement of the problem corresponds to the task of dismantling radiation hazardous objects. The criterion corresponds to the performer's dose load when performing a set of works in conditions of increased radiation. Precedence conditions typical for engineering applications are considered. In an extreme task, the starting point, the order of the tasks and the specific trajectory of the process are optimized.

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